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Mathematical Model for the Hydraulically Driven Spatial Mechanisms

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Abstract

In this work, the realization of the dynamic study for the hydraulically driven mechanism is proposed, by using the equations formulated by Kane, for the study of the general dynamics of the mechanical systems. An original algorithm elaborated on the basis of the mathematical model used for the dynamic study of the hydraulically driven mechanisms with rigid elements and the determination of the inertia matrix in respect with the centre of mass are presented.

Key words: *dynamic direct study, dynamic linear study, hydraulically driven mechanism.*

Introduction

According to the Kane movement equations, [1], the dynamics of a spatial mechanism made up of rigid cinematic elements, is described by the following mathematical model:

$$[M]\cdot[\ddot{q}]+[N_1]\cdot[\dot{q}]+[N_2]\cdot[\dot{q}]+[B]^T\cdot[\lambda]=[Q] \tag{1}$$

$$[M]=\sum_k m_k\left[v_w^k\right]\cdot\left[v_w^k\right]^T+\sum_k\left[\omega^k\right]\cdot\left[J^{k,0}\right]\cdot\left[\omega^k\right]^T \tag{2}$$

$$[N_1]=\sum_k m_k\left[v_w^k\right]\cdot\left[\dot{v}_w^k\right]^T+\sum_k\left[\omega^k\right]\cdot\left[J^{k,0}\right]\cdot\left[\dot{\omega}^k\right]^T \tag{3}$$

$$[N_2]=\sum_k\left[\omega^k\right]\cdot\left[\Omega^{k,0}\right]^T\cdot\left[J^{k,0}\right]\cdot\left[\omega^k\right]^T \tag{4}$$

$$\left[v_w^k\right]=[W]\cdot\left[v^k\right] \tag{5}$$

$$[W]^T\cdot[\dot{q}]=[\omega] \tag{6}$$

$$\left[J^{k,0}\right]=\left[S^{k,0}\right]^T\left[J^k\right]\cdot\left[S^{k,0}\right] \tag{7}$$

The elaboration of the mathematical model is realized by using the algorithm made up of the following steps:

- *The structural analysis of the mechanism*. Based on the cinematic scheme we can determine the structure of the mechanism hydraulically driven, classically, according to the methodology detailed in [7].

- *The settlement of the ramified mechanism*.

The Settlement of the Replacement Cinematic Chains of the Hydraulic Motors

The Substitution Cinematic Chain of the Linear Hydraulic Motor

The linear hydraulic motor with an unilateral bar and with a double effect, or the linear hydraulic motor with an unilateral bar and with a simple effect, is substituted by a cinematic chain made up of two rigid elements, connected between them by a translation couple of V^{th} class [3]. The linear hydraulic motor with a bilateral bar and with a double effect, , or the linear hydraulic motor with a bilateral bar and with a simple effect, , is replaced by a cinematic chain made up of two rigid cinematic elements, connected between them by a translation couple of V^{th} class [5].

The settlement of the mechanism which is not ramified. The transformation algorithm is presented in [7].The determination of the speed and of the acceleration of some points situated on the elements of the mechanism is performed. The speed and the acceleration of a point, reported to the inertia reference point R_0, are calculated in the following way [1, 7]:

- the speed:

$$\vec{v}_k^{\,0} = \left[[\dot{q}]_T^T \left[v_T^k \right] + [\omega]^T \left[v^k \right] \right] \cdot \left\{ \vec{i}^{\,0} \right\} \tag{8}$$

or, using the matrix $[W]$, the relation (8) can be written under the equivalent form:

$$\vec{v}_k^{\,0} = \left[[\dot{q}]_T^T \left[v_T^k \right] + \{q\}^T [W] \cdot \left[v^k \right] \right] \cdot \left\{ \vec{i}^{\,0} \right\} \tag{9}$$

- the acceleration:

$$\vec{a}_k^{\,0} = \left[[\ddot{q}]_T^T \left[v_T^k \right] + [\dot{q}]_T^T \left[\dot{v}_T^k \right] + [\dot{\omega}]^T \left[v^k \right] + [\omega]^T \left[\dot{v}^k \right] \right] \cdot \left\{ \vec{i}^{\,0} \right\} \tag{10}$$

The determination of the Jacobean for the constraint equations:

$$[B] \cdot [\dot{q}] + [b] = [0] \tag{11}$$

The determination of the vector for the generalized force:

$$[Q] = \sum_{k=1}^{n} \left[v_w^k \right] \cdot \left[F_{A,k} \right] + \sum_{k=1}^{n} \left[\omega^k \right] \cdot \left[M_{A,k} \right] \tag{12}$$

The calculation of the friction force associated to the linear hydraulic motor [5]:

$$F_{fMHL} = F_{fv} + F_{fpc} + F_{ftc} \tag{13}$$

The calculation of the viscous friction force between the piston and the cylinder. In this case of flowing, through the slot, it is calculated by using the relation [5]:

$$v_u(u) = -\frac{\Delta p}{2\mu b_p}\left(u^2 - u\delta\right) + c\dot{q}_t\left(1 - \frac{u}{\delta}\right); \qquad \Delta p = p_i - p_e \tag{14-15}$$

The tangential viscous friction force on the surface unit is calculated by the relation (16), [5], and by the substitution of the relation (14) in the relation (16) we get:

$$\tau(u) = \frac{\partial v_u(u)}{\partial u} \;,\; \tau(u) = \frac{\Delta p(2u-\delta)}{2\mu b_p} + c_s \frac{\dot{q}_t}{\delta} \tag{16-17}$$

The surface on which the viscous friction force reacts, has a cylindrical form [7]. Its area is equal to (18) below, and the viscous friction force is calculated by using (19):

$$A(u) = 2\pi\left(\frac{D_p}{2} + u\right)b_p \;,\quad F_{fv} = \int_0^{\delta} \tau(u)A(u)du \tag{18-19}$$

By replacing the relations (17) and (18) in the relation (19) we get the calculation relation:

$$F_{fv} = \frac{\pi\left[\Delta p \delta^3 + 6c_s \mu b_p \left(\delta + D_p\right)\cdot \dot{q}_t\right]}{6\mu} \tag{20}$$

The calculation of the Colombian friction force between the piston and the cylinder:

$$F_{fpc} = \mu_{f1}\pi D_p z_p b_{gp} \Delta p \;,\; F_{ftc} = \mu_{f2}\pi d_t z_t b_{gt} p_e \tag{21-22}$$

The friction force associated to the linear hydraulic motor. The following expression of the friction force associated to the linear hydraulic motor results:

$$F_{fMHL} = f_{f,1}\dot{q}_t + f_{f,0}\,; \quad f_{f,1} = \pi b_p c_s \left(\delta + D_p\right); \tag{23-24}$$

$$f_{f,0} = \frac{\pi\left(\Delta p \delta^3 + 6\mu_{f1} D_p z_p b_{gp} \mu \Delta p + 6\mu_{f2} d_t z_t b_{gt} \mu p_e\right)}{6\mu} \tag{25}$$

$$F_{pe} = p_e A_e; \qquad F_{pr} = p_r A_r \tag{26-27}$$

The calculation of the mass features of the cinematic elements from the structure of the mechanism. These elements take into consideration the existence of the mineral oil inside the hydraulic motor and they allow the consideration of the influence for the mineral oil on the dynamics of the mechanism.

The Inertia Matrixes Associated to the Bodies of the Extension and Retraction Spaces

For a geometrical variety of material volume type, reported to a Cartesian system R of axes, the symmetrical matrix is defined by the inertia tensor by:

$$[J]_R = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix};\; J_{12} = J_{21}\,; J_{13} = J_{31}\;;\; J_{23} = J_{32} \tag{28-30}$$

Considering the density as being expressed by a continuous function, the components of the inertia matrix calculated, reported to the Cartesian system of axes, the following relations are given, [9,10]:

$$J_{11} = \iiint_{VM} \rho_u(x,y,z)\cdot\left(y^2+z^2\right)dxdydz,\;\; J_{22} = \iiint_{VM} \rho_u(x,y,z)\cdot\left(x^2+z^2\right)dxdydz \tag{31-32}$$

$$J_{33} = \iiint_{VM} \rho_u(x,y,z)\cdot\left(x^2+y^2\right)dxdydz,\; J_{12} = J_{21} = \iiint_{VM} \rho_u(x,y,z)\cdot\left(x\cdot y\right)dxdydz \tag{33-34}$$

$$J_{13} = J_{31} = \iiint\limits_{VM} \rho_u(x,y,z)\cdot(x\cdot z)dxdydz \text{ , } J_{23} = J_{32} = \iiint\limits_{VM} \rho_u(x,y,z)\cdot(y\cdot z)dxdydz \quad (35\text{-}36)$$

Based on the relations (31), (32), …, (36), we can determine the inertia matrixes for the bodies associated to the extension and retraction spaces, matrixes that are calculated reported to their mass centre.

The Inertia Matrix of the Body for the Extension Space, Calculated Reported to the Mass Centre

The body is presented in the [8]; we consider the system of axes oriented as it is in the figure 1, with the origin placed in the mass centre. The inertia matrix is the following:

$$J_e = \begin{bmatrix} J_{11e} & 0 & 0 \\ 0 & J_{22e} & 0 \\ 0 & 0 & J_{33e} \end{bmatrix} \quad (37)$$

Its components being calculated by:

$$J_{11e} = j_{11e,1}\cdot q_t + j_{11e,0}; \quad j_{11e,1} = \frac{1}{32}\pi D_p^4 \rho_u l_{e,1}; \quad j_{11e,0} = \frac{1}{32}\pi D_p^4 \rho_u l_{e,0}; \quad (38\text{-}40)$$

$$J_{22e} = j_{22e,3}\cdot q_t^3 + j_{22e,2}\cdot q_t^2 + j_{22e,1}\cdot q + j_{22e,0}; \quad j_{22e,3} = \frac{1}{48}\pi D_p^2 \rho_u l_{e,1}^3; \quad (41\text{-}42)$$

$$j_{22e,2} = \frac{1}{16}\pi D_p^2 \rho_u l_{e,0} l_{e,1}^2; \quad j_{22e,1} = \frac{1}{64}\pi D_p^2 \rho_u l_{e,1}\left(D_p^2 + 4l_{e,0}^2\right); \quad (43\text{-}44)$$

$$j_{22e,3} = \frac{1}{192}\pi D_p^2 \rho_u l_{e,0}\left(3D_p^2 + 4l_{e,0}^2\right); \quad J_{22e} = J_{33e} \quad (45\text{-}46)$$

The Inertia Matrix of the Body Associated to the Retraction Space, Calculated Reported to the Mass Centre

The body is presented in the figure 1; we consider the system of axes as being oriented as in the figure, having the origin placed in the mass centre. The inertia matrix has the form:

$$J_r = \begin{bmatrix} J_{11r} & 0 & 0 \\ 0 & J_{22r} & 0 \\ 0 & 0 & J_{33r} \end{bmatrix} \quad (47)$$

Its components being calculated by:

$$J_{11r} = j_{11r,1}\cdot q_r + j_{11r,0}; \ j_{11r,1} = \frac{1}{32}\pi\rho_u l_{r,1}\left(D_p^4 - d_t^4\right); \ j_{11r,0} = \frac{1}{32}\pi\rho_u l_{r,0}\left(D_p^4 - d_t^4\right); \ (48\text{-}50)$$

$$J_{22r} = j_{22r,3}\cdot q_t^3 + j_{22r,2}\cdot q_t^2 + j_{22r,1}\cdot q_t + j_{22r,0}; \quad j_{22r,3} = \frac{1}{48}\pi\rho_u l_{r,1}^3\left(D_p^2 - d_t^2\right); \quad (51\text{-}52)$$

$$j_{22r,2} = \frac{1}{16}\pi\rho_u l_{r,0} l_{r,1}^2\left(D_p^2 - d_t^4\right); \quad j_{22r,1} = \frac{1}{64}\pi\rho_u l_{r,1}\left(D_p^2 + 4l_{r,0}^2 + d_t^2\right)\left(D_p^2 - d_t^2\right); \quad (53\text{-}54)$$

$$j_{22r,0} = \frac{1}{192}\pi\rho_u l_{r,0}\left(3D_p^2 + 4l_{r,0}^2 + 3d_t^2\right)\left(D_p^2 - d_t^2\right); \quad J_{22r} = J_{33r} \quad (55\text{-}56)$$

Mass Properties of the Equivalent Body

The mass of the equivalent body is equal to the sum: the mass of the body associated to the extension volume, the mass of the body associated to the retraction volume and the mass of the real body of the hydraulic motor:

$$m_e = m_{ue} + m_{ur} + m_c = m_{e,0} + m_{e,1} q_t \, ; m_{e,0} = m_{ue,0} + m_{ur,0} + m_c \, ; m_{e,1} = m_{ue,1} + m_{ur,1} \quad (57\text{-}59)$$

The Weight of the Equivalent Body

It is equal to the sum of the weights: the body associated to the extension volume, the body associated to the retraction volume and the real body of the hydraulic motor. The weight of the body associated to the extension volume:

$$G_{ue} = V_{ue}\gamma_u = g_{ue,0} + g_{ue,1} q_t \, ; \;\; g_{ue,0} = v_{ue,0}\gamma_u = \gamma_u l_{e,0} A_e \, ; \;\; g_{ue,1} = v_{ue,1}\gamma_u = \gamma_u l_{e,1} A_e \quad (60\text{-}62)$$

The weight of the body associated to the retraction volume:

$$G_{ur} = V_{ur}\gamma_u = g_{ur,0} + g_{ur,1} q_t \, ; \;\; g_{ur,0} = v_{ur,0}\gamma_u = \gamma_u l_{r,0} A_r \, ; \;\; g_{ur,1} = v_{ur,1}\gamma_u = \gamma_u l_{r,1} A_r \quad (63\text{-}65)$$

The weight of the equivalent body:

$$G_e = G_{ue} + G_{ur} + G_c = g_{e,0} + g_{e,1} q_t \, ; \quad (66)$$

$$g_{e,0} = g_{ue,0} + g_{ur,0} + G = \gamma_u \left(l_{r,0} A_r + l_{e,0} A_e \right) + G_c \quad (67)$$

$$g_{e,1} = g_{ue,1} + g_{ur,1} = \gamma_u \left(l_{e,1} A_e + l_{r,1} A_r \right) \quad (68)$$

The Application Point of the Equivalent Weight Vector

Reported to this system of axes we have:

o the position vector of the weight force for the body associated to the extension volume:

$$\left|\vec{r}_{ue}\right| = \frac{\left|\vec{r}_{A0D0}\right| + \left|\vec{r}_{A0F0}\right|}{2}; \quad \left|\vec{r}_{A0D0}\right| = q_t + L_{r0} + L + L_{e0} \, ; \quad \left|\vec{r}_{A0F0}\right| = L_{r0} + L \, ; \quad (69\text{-}71)$$

$$\vec{r}_{ue} = \left(r_{ue,0} + r_{ue,1} q_t \right) \cdot \vec{i}_1 \, ; \quad r_{ue,0} = L + L_{r0} + \frac{L_{e0}}{2} \, ; \quad r_{ue,1} = \frac{1}{2} \quad (72\text{-}74)$$

o the position vector of the weight force for the body associated to the retraction volume:

$$\left|\vec{r}_{ur}\right| = \frac{\left|\vec{r}_{A0B0}\right| + \left|\vec{r}_{A0C0}\right|}{2}; \quad \left|\vec{r}_{A0B0}\right| = q_t + L_{r0} + L + L_{e0} \, ; \quad \left|\vec{r}_{A0C0}\right| = q_t \, ; \quad (75\text{-}77)$$

$$\vec{r}_{ur} = \left(r_{ur,0} + r_{ur,1} q_t \right) \cdot \vec{i}_1 \, ; \quad r_{ur,0} = \frac{L + L_{r0} - g_p}{2} \, ; \; r_{ur,1} = \frac{1}{2} \quad (78\text{-}80)$$

o the position vector of the weight force for the real body of the hydraulic motor :

$$\vec{r}_c = r_{c,1} \cdot \vec{i}_1 + r_{c,2} \cdot \vec{i}_2 \quad (81)$$

Using the vectors calculation relation for the vector of the application point, we can determine the resultant of a system of parallel vectors, [7]:

$$\vec{r} = \frac{\sum_K \vec{r}_K \cdot \vec{G}_K}{\vec{G}_K} \tag{82}$$

Based on the relation (82), knowing the vectors of the weight forces, we can write:

$$\vec{r}_e = r_{e,1} \cdot \vec{i}_1 + r_{e,2} \cdot \vec{i}_2 ; \quad r_{ue} = \left|\vec{r}_{ue}\right| ; \quad r_{ur} = \left|\vec{r}_{ur}\right| ; \quad r_{C1} = \vec{r}_C \cdot \vec{i}_1 ; \tag{83-84}$$

$$r_{e,1} = \frac{r_{ue} G_{ue} + r_{ur} G_{ur} + r_{c,1} G_c}{G_e} = \frac{r_{e1,2} q_t^2 + r_{e1,1} q_t + r_{e1,0}}{g_{e,0} + g_{e,1} q_t} ; \; r_{e1,2} = r_{ue,1} g_{ue,1} + r_{ur,1} g_{ur,1} \tag{85-87}$$

$$r_{e1,1} = r_{ue,1} g_{ue,0} + r_{ur,0} g_{ur,1} + r_{ur,0} g_{ur,1} + r_{ur,1} g_{ur,0} ; \; r_{e1,0} = r_{ue,0} g_{ue,0} + r_{ur,0} g_{ur,0} + r_{c,1} G_c \tag{88-89}$$

$$r_{e,2} = \frac{r_{c,2} G_c}{G_e} ; \quad r_{e,2} = \frac{r_{e2,0}}{g_{e,0} + g_{e,1} q_t} ; \quad r_{e2,0} = r_{c,2} G_c \tag{90-92}$$

The Inertia Matrix of the Equivalent Body, Calculated Reported to the Mass Centre

The matrixes are calculated reported to the mass centre of the equivalent body using the Steiner theorem, [3]. We are interested in the manner of the distance vectors: $\vec{d}_{ue}$, $\vec{d}_{ur}$ şi $\vec{d}_e$. Because the vectors of the application points corresponding to the forces are:

$$\vec{r}_e = r_{e,1} \cdot \vec{i}_1 + r_{e,2} \cdot \vec{i}_2 ; \quad \vec{r}_c = r_{c,1} \cdot \vec{i}_1 + r_{c,2} \cdot \vec{i}_2 ; \quad \vec{r}_{ur} = r_{ur} \cdot \vec{i}_1 ; \quad \vec{r}_{ue} = r_{ue} \cdot \vec{i}_1 \tag{93-95}$$

We have:

$$d_C = \left|\vec{d}_C\right| = \sqrt{\left(r_{e,1} - r_{c,1}\right)^2 + \left(r_{e,2} - r_{c,2}\right)^2} ; \quad d_{ur} = \left|\vec{d}_{ur}\right| = \sqrt{\left(r_{e,1} - r_{ur}\right)^2 + \left(r_{e,2}\right)^2} \tag{96-97}$$

$$d_{ue} = \left|\vec{d}_{ue}\right| = \sqrt{\left(r_{e,1} - r_{ue}\right)^2 + \left(r_{e,2}\right)^2} \tag{98}$$

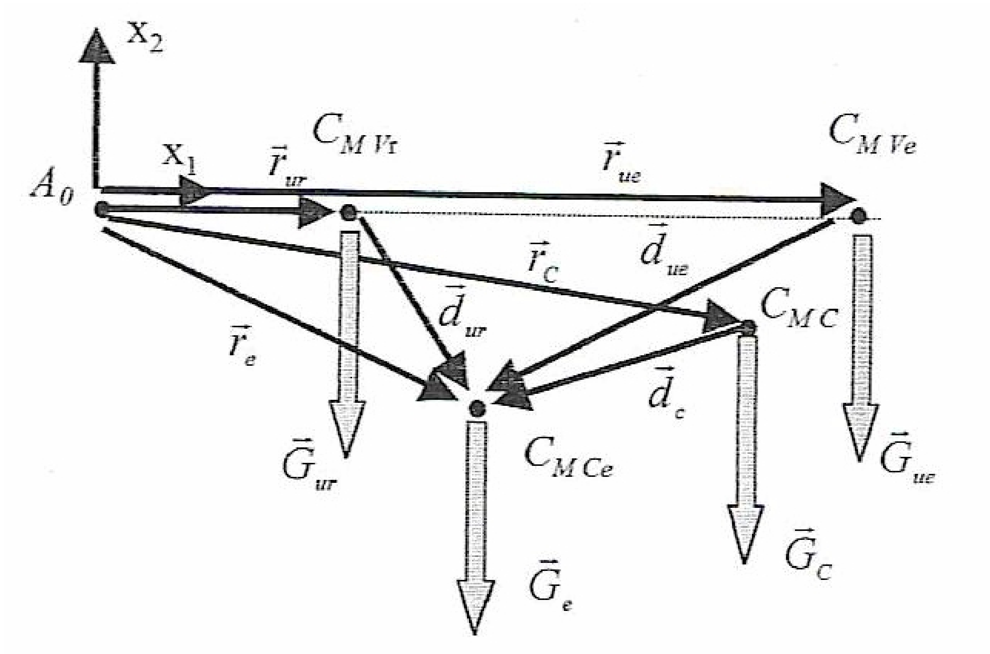

Fig. 1

Conclusions

Having the mathematical model we can write in a programming language (preferably Matlab/Maple), the program which determines under an analytical or numerical form, the friction force associated to the linear hydraulic motor according to the calculation algorithm previously presented. In the same time we can write the program which allows the determination of the inertia matrix associated to the extension body reported to the mass centre.

In the same time, we can write in a programming language (Matlab/Maple), a program which could be used to determine the inertia matrix associated to the volume of the retraction body.

We can write the program that can be used to calculate the mass of the equivalent body. In order to calculate the inertia matrix of the equivalent body, reported to the mass centre, it is necessary to determine the position of this centre. In this aim, we calculate the weight force of the equivalent body, whose application point coincides to the mass centre of the equivalent body. In this case we can write the program that can be used to determine the weight of the equivalent body.

In the same time, we can write a program that could allow the determination of the inertia matrix of the equivalent body.

Symbols

m_k – the mass of the cinematic element k;
[q] – the vector of the generalized co-ordinates;
[λ] – the vector of the Lagrange multiplier;
[B] – the Jacoben of the system of constraint equations;
[Q] – the vector of the generalized forces;
[ω] – the vector of the generalized speeds;
[J^k] – the inertia matrix associated with the cinematic element *k*, calculated by reporting it to the mass centre;
[M] – the mass matrix associated to the system;
[v^k] – the partial speed matrix corresonding to the absolute speed of the cinematic element *k* associated to the vector of the generalized speeds;
[ω^k] – the partial matrix of the angular speed associated to the cinematic element *k*;
[$S^{k,0}$] – the absolute rotary matrix of the own reference point attached to the cinematic element *k*;
[$\Omega^{k,0}$] – the antisystemic matrix associated to the co-ordinates of the absolute angular speed vector of the cinematic element *k*;
[$F_{A,k}$] – the resultant of the exterior forces that react on the cinematic element *k*;
[$M_{A,k}$] – the exterior couples that react on the cinematic element *k*;
F_{fv} – the viscous friction force between the piston and the cylinder;
F_{fpc} – the Colombian friction force between the piston and the cylinder;
F_{ftc} – the Colombian friction force between the bar and the sealing head;
Δp – the drop pressure on the slot;
p_i – the pressure of the oil at the entrance in the slot;
P_e – the pressure of the oil at the exit from the slot;
μ – the dyhamic viscosity of the oil;
D_p – the diameter of the piston;
b_p – the width of the piston;
δ – the thickness of the oil throught the slot;
μ_{f1} – the Colombian friction coefficient between the piston and the cylinder;
z_p – the number of the piston air sealing;
b_{gp} – the width of the piston air sealing;
μ_{f2} – the Colombian friction coefficient between the bar and the head;
z_t - the number of the piston air sealing;
b_{gt} – the width of the piston air sealing;

d_t – the diameter of the bar;
p_e – the pressure from the annular chamber of the hydraulic motor;
C_{MVr} – the mass centre of the body associated to the reaction volume;
C_{Mve} – the mass centre of the body associated to the extension volume;
C_{MC} – the mass centre of the real body of the hydraulic motor.

References

1. Buculei, M. – *Metode de calcul in analiza mecanismelor cu bare,* Editura Scrisul Românesc, Craiova, 1986.
2. Buculei, M., Marin, M. – *Elemente de mecanică tehnică*, Editura Universitaria, Craiova, 1994.
3. De Jalon, J.G., Bayo, E. – *Kinematic and Dynamic Simulation of Multibody Systems*, Springer-Verlang, 1994.
4. Amirouche, F.M.L. – *Computational methods in multibody dynamics*, Pretince Hall, England Cliffs, New Jersey,1992.
5. Popescu, I. – *Mecanisme. Teorie şi aplicații,* Reprografia Universității din Craiova, 1991.
6. Viersma , T.J. – *Analysis, synthesis and design of hydraulic servosystems and pipelines*, Delft University of Technology, Faculty of Mechanical Engineering, Holland, 1990.

Model matematic pentru mecanisme spațiale acționate hidraulic

Rezumat

În lucrare se propune realizarea studiului dinamic al mecanismelor acționate hidraulic, prin utlizarea ecuațiilor formulate de Kane, la studiul dinamicii generale a sistemelor mecanice. Se prezintă un algoritm original elaborat pe baza modelului matematic utilizat pentru studiul dinamic al mecanismelor acționate hidraulic cu elemente rigide şi determinarea matricii de inerție în raport cu centrul de masă, calculul masei corpului echivalent şi determinarea punctului de aplicație a forței de greutate a corpului echivalent.